

**THE DETERMINANT FORMULATION
AND NEWTON-LIKE METHODS
FOR THE NUMERICAL SOLUTION
OF TWO-POINT BOUNDARY
VALUE PROBLEMS**

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The determinant formulation and
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This dissertation shows that discretizations of the integral
equation formulation of the two-point boundary value problem

$$(py')' + qy = f(t, y(t)), \quad 0 < t < 1,$$

$$a_{11}y(0) - a_{12}y'(0) = c, \quad b_{21}y(1) + b_{22}y'(1) = d,$$

after a simple transformation, called the determinant formulation,
have the desirable property of being tridiagonal and in general of
"positive type". By the determinant formulation Newton-like methods
are constructed for the numerical solution of such problems. These
methods constitute a homotopic connection between the method of
successive approximations and Newton's method over a set of positive
integers. Computation in parallel is possible.
Determinant formulation and Newton-like methods for discrete
tridiagonal problems and integral equations with a semidegenerate
kernel are also given.

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THE DETERMINANT FORMULATION AND NEWTON-LIKE METHODS FOR THE NUMERICAL SOLUTION OF TWO-POINT BOUNDARY VALUE PROBLEMS

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The dissertation introduced and summarized here is based on the following papers:

- I. The determinant formulation of discrete and continuous two-point boundary value problems, Report LUNFD6/(NFIP-3001)/1-144/(1976), Department of Computer Sciences, Lund University, Lund, Sweden.
- II. Newton-like methods for two-point boundary value problems, BIT 16 (1976), 1-12, 346.
- III. Newton-like methods for the numerical solution of discrete and continuous two-point boundary value problems, Report LUNFD6/(NFIP-3002)/(1977), Department of Computer Sciences, Lund University, Lund, Sweden.
- IV. Newton-like methods for integral equations with semi-degenerate kernels, Report LU/CS 75-10, 1975, Department of Computer Sciences, Lund University, Lund, Sweden.

1. FORMULATION OF TWO-POINT BOUNDARY VALUE PROBLEMS

We will consider the following boundary value problem

$$(1) \quad \begin{cases} Ly = (py')' + qy = f(t, y(t)), & 0 < t < 1, \\ B_1 y = a_{11}y(0) - a_{12}y'(0) = c, \\ B_2 y = b_{21}y(1) + b_{22}y'(1) = d. \end{cases}$$

If $\lambda = 0$ is not an eigenvalue of the eigenvalue problem $Ly + \lambda y = B_1 y = B_2 y = 0$ then (1) can be formulated as an equivalent integral equation (Stakgold [9])

$$(2) \quad y(x) = \int_0^1 G(x, t) f(t, y(t)) dt + \frac{cb(x)}{B_1(b)} + \frac{da(x)}{B_2(a)}, \quad 0 \leq x \leq 1,$$

where

$$(3) \quad G(x, t) = \begin{cases} b(x)a(t), & x \geq t, \\ a(x)b(t), & x \leq t, \end{cases}$$

is Green's function for L , subject to the boundary conditions B_1 and B_2 of (1). Green's function for the boundary value problem (1), with $q \neq 0$, is in general not known. However, if we move the term qy to the right hand side, then Green's function for the differential operator of the left hand side will be easy to obtain.

Another formulation of the differential equation (1) is the weak form. There are several, but they all share the following basic principle: The equation is multiplied by test functions $v(x)$ and integrated over the interval $[0, 1]$, to yield

$$\int_0^1 (Ly)v dt = \int_0^1 f v dt.$$

This must hold for each function v in some test space V . The weak formulation can be used to share the regularity requirements between the solution y and the test function v (Strang and Fix [10]).

2. THE DETERMINANT FORMULATION

The integral equation formulation of two-point boundary value problems has not been a popular tool for the numerical solution of these problems. The reason is that the linearization of (2) leads to a linear system with a dense matrix, while the finite-difference approximation of the differential equation in (1) will give a tridiagonal matrix. Paper I shows how the integral equation (2) can be written in a form which gives a tridiagonal matrix. We will call this form the determinant formulation of the boundary value problem (or of the integral equation).

The tridiagonal matrix of the determinant formulation is common to both the continuous problem and the discrete approximation

of the problem. Hence this matrix can be analysed by analytical methods, i.e. by the following eigenvalue problems

$$(4) \quad \begin{cases} Ly + \lambda y = 0 \\ y(0) = y(1) = 0, \end{cases} \quad \begin{cases} Ly + \mu y = 0 \\ B_1 y = B_2 = 0. \end{cases}$$

If $\mu > 0$ paper I shows that the matrix of the determinant formulation is an M-matrix (for the definition of an M-matrix see Ortega [8] p. 108). Paper I gives both $O(h^2)$ and $O(h^4)$ discretizations of the determinant formulation of (1). The discretization at the end-points of the interval appears very natural and we never use any point outside the interval. All discretizations are based on quadrature formulas.

The determinant formulation can be interpreted as special cases of Peano's kernel theorem or of the weak formulation of the problem. It is interesting to note that the finite-difference approximation

$$p\left(\frac{i+\frac{1}{2}}{n}\right)\left[y\left(\frac{i+1}{n}\right) - y\left(\frac{i}{n}\right)\right] - p\left(\frac{i-\frac{1}{2}}{n}\right)\left[y\left(\frac{i}{n}\right) - y\left(\frac{i-1}{n}\right)\right] = \frac{1}{n^2} f\left(\frac{i}{n}, y\left(\frac{i}{n}\right)\right)$$

of the differential equation (1), with $q = 0$, is also the approximation of the integral equation, the determinant formulation, and the weak formulation by the trapezoidal rule and the midpoint rule.

Paper I also shows that the determinant formulation may exist if $\mu = 0$, i.e. when Green's function does not exist.

3. NEWTON-LIKE METHODS

For a Fredholm integral equation of the second kind Brakhage [7] and Atkinson [1], [2] have used an iterative variant of Nyström's method to obtain approximate solutions. Newton's method for the integral equation (2) is

$$(5) \quad \begin{cases} u(x) - \int_0^1 G(x,t)f'_y(t,y(t))u(t)dt = -[y(x) - \int_0^1 G(x,t)f(t,y(t))dt - \frac{cb(x)}{B_1(b)} - \frac{da(x)}{B_2(a)}], & 0 \leq x \leq 1, \\ y := y + u. \end{cases}$$

Let m, n , and r be positive integers and $n = rm$. Then m and n define two grids, G_m and G_n , on $[0,1]$ with equidistant points. The iterative variant of Nyström's method consists of two steps. First the left hand side of (5) is approximated on the coarser grid G_m and the right hand side on G_n . We then obtain a linear system for u on G_m . In the

second step intermediate values of u , on G_n , are found by using the discrete version of (5) as an interpolation formula. We call this procedure a Newton-like method. However, this procedure is very ineffective from a numerical point of view. First the left hand side corresponds to a dense $(m+1) \times (m+1)$ matrix, and second the right hand side involves the computation of sums with $n+1$ terms. The number of operations for this Newton-like method is proportional to n^2 . Thus this method cannot compete with the method of finite-difference approximation of (1).

One of the most important properties of the determinant formulation is that it can easily be shifted from one grid to another and this process is commutative. Thus, if we first apply the determinant formulation to (5) on the grid G_m , then this discretization gives a system with a tridiagonal matrix. The elements of the vector of the right hand side will now contain sums with $2r$ terms. To compute intermediate values of u we apply the determinant formulation to the discrete version of (5) on the grid G_n . Then the intermediate values of u are explicitly defined by this procedure.

The advantage of Newton-like methods is that f'_y is evaluated only on the grid G_m and we only have to solve a tridiagonal system with $m+1$ equations. Furthermore the main part of the algorithm is the evaluation of $2m$ sums with r terms each and these sums can all be evaluated in parallel. These sums are used in both steps of the Newton-like method.

The first step can be interpreted as Newton's method and the second step as the method of successive approximations. Hence the Newton-like method can be considered as a compromise between the method of successive approximations and Newton's method. An optimal choice of m will depend of the amount of work required to compute f'_y , the rate of convergence for different values of m , and the possibility of doing the computations in parallel.

In paper II Newton-like methods for the simple problem

$$\begin{cases} y'' = f(y(t)), & 0 < t < 1, \\ y(0) = a, y(1) = b, \end{cases}$$

is given (see also Aulin [3]), and in paper III Newton-like methods are given for the boundary value problem (1) (see also Aulin [4], [6]). Convergence results are also presented.

4. EXTENSIONS

In paper I the determinant formulation of discrete equations, i.e. a discrete system with a tridiagonal matrix, is given while paper III treats Newton-like methods for discrete equations.

Green's function $G(x,t)$ in (2) is a special case of a class of kernels which are called semidegenerate. Such kernels have the form

$$K(x,t) = \begin{cases} \sum_{\mu=1}^M a_{\mu}(x)b_{\mu}(t), & x > t, \\ \sum_{v=1}^N c_v(x)d_v(t), & x < t. \end{cases}$$

Green's function for two-point boundary value problems of the order $2k$, $k = 1, 2, \dots$ have this form. In paper IV the determinant formulation and Newton-like methods for integral equations with a semidegenerate kernel are defined.

In Aulin [5] Newton-like methods, based on the method in paper II, for the numerical solution of the boundary value problem $u_{xx} + u_{yy} = f(x, y, u(x, y))$, u given on the unit square, are given.

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